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Sharing Water from many Rivers

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Abstract

This paper studies the problem of non-cooperative water allocation between heterogeneous communities embodied in an acyclic network of water sources. The extraction activity of a community has a negative impact on the extraction activity of its direct successors: it reduces the intensity of water flows entering their source, and thus, increase their convex costs of water extraction. We show that the equilibrium profile is unique and may be expressed through complementarity and substitutability effects which characterize the incoming centrality of a community in the network of sources. For each community, the efficient activity is a combination of two opposite network effects, the incoming centrality and the outcoming centrality. Then, the optimal tax rate imposed to a community depends on the network structure, and reflects both the marginal damages and the marginal benefits this community delivers to other communities at the efficient extraction activity profile.

Keywords: flowing water, network of sources, equilibrium effects, efficiency effects, optimal tax.

JEL: A14, H21, Q25.

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1 Introduction

Consider a set of communities naturally distributed across a river basin composed of many rivers. Each community extracts water from its closest river, and the flowing nature of water determines unidirectional dependency among communities. In general, the most disadvantaged participants would be the ones located at the downstream tails of rivers. In river basins, however, convergence of flow may occur in certain locations, and since water flows may have different intensities, the negative effects for downstream participants may be counterbalanced. How to achieve the efficient allocation of water if communities do not cooperate? Do the structure and intensity of water flows between communities matter?

In the classic common property resource dilemma described by Gordon (1954) and Hardin (1960), participants face reciprocal externalities, and there is often a feasible state, based on institutional agreements, that make all the participants and the ecosystem better off (Ostrom, 1990; Ostrom et al., 1994). This cooperative solution, built upon the Coase theorem, supposes that property rights are well-defined. In case of local common property resources, property rights should be exercised collectively by members of small communities (Seabright, 1993). When dealing with flowing water, however, rights owned by the agents are difficult to determine.

Studies on cooperative sharing of water from a river refer to the theory of Absolute Territorial Sovereignity and the theory of Unlimited Territorial Integrity, to define property rights. Under quasi-linear preferences over water and money, the unique distribution that improves individual and social welfare, the Downstream Incremental Distribution, has been found by Ambec and Sprumont (2002). When the river caries pollutants to participants, Ni and Wang (2007) have established two equivalent methods to divide fairly the total river-polluting responsability among the polluters: the Local Responsability Sharing and the Upstream Equal Sharing. More recent contributions include Ambec (2008), Ambec and Ehlers (2008) and Wang (2011).

From a non-cooperative perspective, when there are riparian water rights, the optimal tax plan to achieve the efficient allocation of water may involve different tax rates at different sites throughout the river (Janmaat, 2005). One unifying limit of these approaches, however, is to consider one single river. This present paper deals with the problem of sharing water from many connected rivers. For this purpose, we study a network game[1], in which players have their own source of water and extract the resource individualistically.

Specifically, there is a finite number of heterogeneous communities embodied in a network of water sources. At each source, one or several hydrological flows of a certain intensity may impact the effort necessary to extract a given amount of water. More precisely, the extraction activity of a community has a negative impact on the extraction activity of its direct successors: it reduces the intensity of water flows entering their source, and thus, increase their costs of water extraction. Communities benefit only from their extraction activity, but their marginal costs, strictly positive and increasing in the activity of water extraction, depend both on their own activity and on the activity of their direct predecessors.

Related works include İlkiliç (2010), who studies a water extraction game in bipartite networks, when there are multiples sources, and the users exploit the sources freely. He characterizes the unique equilibrium in terms of the centrality of an agent in the network, as well as the efficient allocation of

[1]See Jackson (2008) for an overview of social networks and economic application with respect to how they influence social and economic activity, and how they can be modeled and analyzed.

water. In his paper, links connect participants (cities) with sources. Two participants interact if they share a source, and each link reflects a quantity of extracted water. Our approach is different: there are property rights over water sources and links connect sources directly. Two participants interact if their sources are hydrologically dependent, and each link denotes an intensity of hydrological influence.

This work is also related to the literature on network games with strategic substitutes (Bramoullé and Kranton, 2007; Ballester and Calvó-Armengol, 2010). In this paper, we chose to model the activity of water extraction from many connected rivers as a means of investigating weighted, directed and acyclic network games, when a community's extraction activity is an imperfect substitute of the activities of its predecessors, and when players have heterogeneous preferences. Our results allow to characterize in terms of the Bonacich centrality vector, a widely used network centrality measure[2], the optimal tax plan to achieve the efficient allocation of water between the communities.

The remainder of the paper is organized as follows. The network of sources is introduced in Section 2. Section 3 defines and studies the game played by the communities. Section 4 characterizes the efficient profile. Section 5 discusses the results. The main proofs are relegated to the appendix.

2 The network of sources

Matrices are represented as bold upper case and vectors as bold lower case. All vectors are column vectors. The transpose of a matrix $\mathbf{M}$ is denoted $\mathbf{M}^T$. Let $\mathbf{I}$ stands for the identity matrix.

[2]For its use in network games, see, e.g., Ballester et al. (2006); Ballester and Calvò-Armengol (2009), Calvó-Armengol et al. (2009) or İlkiliç (2010).

We represent the network formed by the water sources as a graph. The basic notation, some of which we borrow from Godsil and Royle (2001), is as follows.

A *weighted directed graph* G consists of a vertex set $V(G) = \{1, \ldots, N\}$ formed by water sources, an arc set $A(G)$ formed by flows of hydrological influence, where a flow is an ordered pair of distinct sources, and a mapping from the set of flows to a set of intensities $I(A)$. We will use ij to denote a flow directed from source i to j, and ω_{ij} its associated intensity. If ij is a flow, then we say that source j is a successor of source i, or that source i is a predecessor of source j.

The *weighted adjacency matrix* $\mathbf{\Omega}(G)$ of a weighted directed graph G is the non-negative matrix with rows and columns indexed by the sources, such that the ij-entry of $\mathbf{\Omega}$ is equal to ω_{ij} if $ij \in A$, and 0 otherwise. The *unweighted adjacency matrix* $\mathbf{M}(G)$ of a weighted directed graph G is the Boolean matrix with rows and columns indexed by the sources, such that the ij-entry of $\mathbf{M}$ is equal to 1 if $ij \in A$, and 0 otherwise.

As water flows from a river, because of gravity, always go from up to down, we assume that flows of hydrological influence may form directed paths, but not directed cycles. A directed path of length r is a sequence of $r+1$ distinct sources connected by flows corresponding to the order of the sources in the sequence. A directed cycle is a directed path whose first and last sources are the same. A *weighted directed acyclic graph* (wDAG) is a weighted directed graph with no directed cycles.

The wDAG $G = \langle V, A, I(A) \rangle$ with weighted adjacency matrix $\mathbf{\Omega}$ and unweighted adjacency matrix $\mathbf{M}$ is called the *network of sources*.

3 The game played by the communities

There are N communities, and the set of communities is $\mathcal{N} = \{1, ..., N\}$. Communities are embodied in a network of sources, and each community extracts water from its own single source. For all community i, let a_i denote the extraction activity, $q_i(.)$ an increasing twice differentiable strictly convex cost function, and p_i the strictly positive benefit parameter. Letting $\mathbf{a}$ denote the N-vector of extraction activities, the maximum utility, U_i, can then be determined by solving, for all i,

$$U_i(\mathbf{\Omega}, \mathbf{a}) = \text{Max } p_i a_i - q_i(a_i + \sum_{j:j \neq i} \omega_{ji} a_j).$$

The strategy of utility-maximizing water extraction activities involves a simultaneous-move game. For all community, utility maximization occurs where the marginal benefit of extraction is equal to its marginal cost. Let a_i^* denote the value of extraction activity such that $p_i = q_i'(a_i^*)$. We call a_i^* the *equilibrium peak* of community i. We note $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ the game played by the communities, where $\mathbf{a}^*$ is the N-vector of equilibrium peaks.

We suppose that, for all i, $q_i'(0) < p_i < q_i'(\infty)$. As shown by the following figure, this assumption guarantees, for all community, the existence of a positive and finite equilibrium peak. Note that the marginal utility is decreasing. It is positive until a certain value of extraction activity, and becomes negative above this point.

Given $\mathbf{\Omega}$, the best response $\hat{a}_i$ of community i to $\mathbf{a} \in (\mathbb{R}_+)^N$ is such that,

$$\hat{a}_i = \begin{cases} a_i^* - \sum\limits_{j:j \neq i} \omega_{ji} a_j, & \text{if } \sum\limits_{j:j \neq i} \omega_{ji} a_j \leq a_i^*. \\ 0, & \text{otherwise.} \end{cases}$$

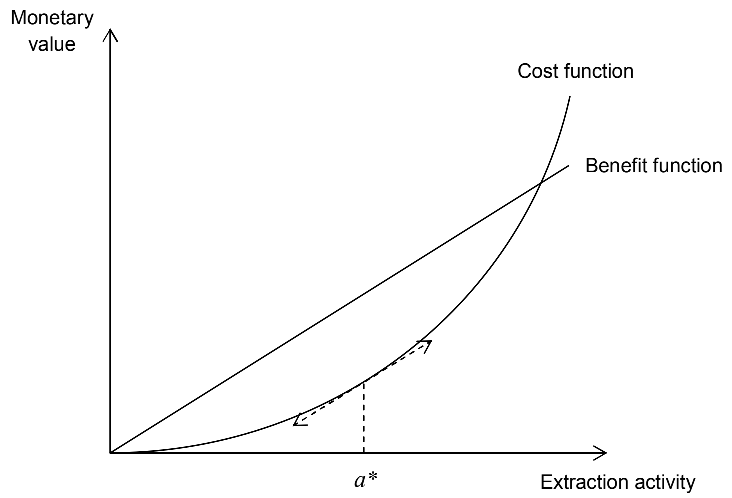

Figure 1: Utility-maximization of water extraction activity

For all community i, the activity of water extraction is positive as long as $\sum_{j:j\neq i}\omega_{ji}a_j$ is less than a_i^*. If the impact of its predecessors' extraction activity is more than a_i^*, the community extracts no water. If the impact is less than a_i^*, the community extracts water up to the point where the value of extraction activity equals a_i^*. It follows that a water extraction profile $\hat{\mathbf{a}} \in (\mathbb{R}_+)^N$ is a Nash equilibrium of the game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ if, and only if, for all i,

$$\hat{a}_i = \max\{0, a_i^* - \sum_{j:j\neq i}\omega_{ji}\hat{a}_j\}.$$

This class of Nash equilibria has been studied by several authors. When networks are unweighted and undirected, and when individual preferences are homogeneous, Bramoullé and Kranton (2007) have shown that multiple equilibria is the rule. When networks are weighted and when individuals have heterogeneous preferences, Ballester and Calvò-Armengol (2010) have found that there is a unique Nash equilibrium if the spectral radius of the weighted adjacency matrix is low enough. This condition suffices to guarantee that best response functions are contraction mappings.

In this present paper, we design a step-by-step proof to show that the game played by the communities always admits a unique Nash equilibrium. Our proof is constructive, since it makes the equilibrium profile to appear naturally. The algorithm is formally established in the Appendix; the intuition is as follows.

It is known that every directed acyclic graph has at least one vertex with no in-coming arcs and one vertex with no out-coming arcs.[3] Thus, every network game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ contains at least one community with no predecessor. We refer to these participants as the *source players of rank 1* and their set is denoted C_1. They have a unique best response which is to extract an amount of water equal to their equilibrium peak. That is, for all $i \in C_1$,

$$\hat{a}_i = a_i^*.$$

Therefore, the equilibrium peak of the other players is *modified*, i.e., for all $j \notin C_1$,

$$a_j^1 = \begin{cases} a_j^* - \sum_{i \in C_1} \omega_{ij} a_i^*, & \text{if } a_j^* \geq \sum_{i \in C_1} \omega_{ij} a_i^*. \\ 0, & \text{otherwise.} \end{cases}$$

and therefore,

$$\hat{a}_j = \max\{0, a_j^1 - \sum_{k \notin C_1} \omega_{kj} \hat{a}_k\}.$$

Next, consider the subgame obtained after deletion of all the source players of rank 1 and their out-coming flows. Since any subgraph of an acyclic graph is also acyclic, a new set of source players appears. Let us call them the *source players of rank 2*. Their set is denoted C_2. Since they are also

[3] See, e.g., Godsil and Royle (2001, p. 337).

source players of rank 1 in this subgame, they have a unique best response which is to extract an amount of water equal to their modified equilibrium peak. That is, for all $j \in C_2$,

$$\hat{a}_j = \max\{0, a_j^* - \sum_{i:i \neq j} \omega_{ij} \hat{a}_i\} = a_j^1.$$

Therefore, the equilibrium peak of the other players is modified again, i.e., for all $k \notin C_1 \cup C_2$,

$$a_k^2 = \begin{cases} a_k^1 - \sum_{j \in C_2} \omega_{jk} a_j^1, & \text{if } a_k^1 \geq \sum_{j \in C_2} \omega_{jk} a_j^1. \\ 0, & \text{otherwise.} \end{cases}$$

and it follows that

$$\hat{a}_k = \max\{0, a_k^2 - \sum_{l \notin C_1 \cup C_2} \omega_{lk} \hat{a}_l\}.$$

Then, repeat the same procedure until the last set of source players is considered. If the length of the longest directed path in G is p, the procedure is repeated $p+1$ times. Let us denote by C_{p+1} the last set of source players. Since they are also source players of rank 1 in the final subgame, they have a unique best response which is to play their p-times modified equilibrium peak. Therefore, each player has a unique best response to his predecessors' action, and we may state the following result.

Theorem 1. *The network game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ admits a unique Nash equilibrium.*

It should be noted that a more "traditional" proof of Theorem 1, based on the spectral radius of the weighted adjacency matrix, may also be established.

Remark 1. By Theorem 1 of Nicholson (1975, p. 186), a graph G is acyclic if, and only if, its unweighted adjacency matrix $\mathbf{M}(G)$ is nilpotent. Therefore, its weighted adjacency matrix $\mathbf{\Omega}(G)$ is also nilpotent by Lemma A1 (see the Appendix). A square matrix is nilpotent if, and only if, all of its eigenvalues are zero. Hence, a graph G is acyclic if, and only if, the spectral radius of its weighted adjacency matrix is equal to zero. By Proposition 2 of Ballester and Calvó-Armengol (2010, p. 403), we conclude that the network game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ admits a unique Nash equilibrium.

Let $(Q')^{-1}(\mathbf{p}) = ((q'_k)^{-1}(p_k))_k$. Applying the same induction reasoning as in Theorem 1, we obtain the following result.

Theorem 2. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ be a network game and p the length of the longest directed path in $\mathbf{\Omega}$. If for all i, $a_i^* - \sum_{j:j\neq i} \omega_{ji}\hat{a}_j \geq 0$, the Nash equilibrium profile is*

$$\hat{\mathbf{a}} = \mathbf{a}^* + \sum_{k=1}^{\lfloor p/2 \rfloor} (\mathbf{\Omega}^T)^{2k}\mathbf{a}^* - \sum_{k=1}^{\lceil p/2 \rceil} (\mathbf{\Omega}^T)^{2k-1}\mathbf{a}^*,$$

where

$$\mathbf{a}^* = (Q')^{-1}(\mathbf{p}).$$

The first summation denotes the total weight of even directed paths that end at the corresponding vertex in G, and the second summation denotes the total weight of odd directed paths that end at it, where directed paths that start from j are weighted by a_j^*.

The first sum tells that the equilibrium extraction activity of a community is positively related with the weight of even length directed paths that finish at it. The water flows on links which have an odd number of sources between them are strategic complements. In contrast, the negative sign on the second summation means the equilibrium extraction activity of a community is negatively related with the weight of odd length directed paths that finish at it. The water flows on links which have an even number of sources are strategic substitutes. These incoming "substitutability-complementarity" effects are given by

$$\mathbf{E}^{\mathrm{eq}} = \sum_{k=1}^{\lfloor p/2 \rfloor} (\boldsymbol{\Omega}^T)^{2k} - \sum_{k=1}^{\lceil p/2 \rceil} (\boldsymbol{\Omega}^T)^{2k-1},$$

which may be called the *equilibrium effects* matrix. Related results include Ballester and Calvò-Armengol (2010) and İlkiliç (2010).

To conclude this section, it should be noted that Theorem 2 characterizes the equilibrium profile in terms of the centrality of a community in the network of sources, and is valid for all kind of pure-strategy Nash equilibria.

Remark 2. Let $\boldsymbol{\Lambda}$ be the (non-negative) weighted adjacency matrix of a wDAG in which z is the length of the longest directed path. By Lemma A1, $\boldsymbol{\Lambda}$ is nilpotent and we have the following algebraic identity,

$$\sum_{k=0}^{z} \boldsymbol{\Lambda}^k = \mathbf{I} + \sum_{k=1}^{z} \boldsymbol{\Lambda}^k = (\mathbf{I} - \boldsymbol{\Lambda})^{-1}$$

which shows that $\mathbf{I} - \boldsymbol{\Lambda}$ is invertible and has a non-negative inverse. Let $\mathbf{e} >> 0$ be a weight vector. Following Bonacich (1987), the vector

$$\mathbf{b}^+(\boldsymbol{\Lambda}, \mathbf{e}) = (\mathbf{I} - \boldsymbol{\Lambda})^{-1}\mathbf{e} = \mathbf{e} + \sum_{k=1}^{z} \boldsymbol{\Lambda}^k \mathbf{e}$$

is called the *outcoming weighted Bonacich centrality* measure applied to $\mathbf{e}$. Its entry b_i^+ denotes the total weight of all directed paths in G that start from i, where all directed paths that end at j are weighted by e_j. Therefore, the vector

$$\mathbf{b}^-(\boldsymbol{\Lambda}, \mathbf{e}) = (\mathbf{I} - \boldsymbol{\Lambda}^T)^{-1}\mathbf{e} = \mathbf{e} + \sum_{k=1}^{z} (\boldsymbol{\Lambda}^T)^k \mathbf{e}$$

is called the *incoming weighted Bonacich centrality* measure applied to $\mathbf{e}$. Its entry b_i^- denotes the total weight of all directed paths in G that end at i, where all directed paths that start from j are weighted by e_j.

Following these definitions, as well as the definition of series where terms alternate signs, we may define the vector

$$\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Lambda}, \mathbf{e}) = (\mathbf{I} + \mathbf{\Lambda})^{-1}\mathbf{e} = \mathbf{e} + \sum_{k=1}^{z} (-1)^k \mathbf{\Lambda}^k \mathbf{e},$$

that is,

$$\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Lambda}, \mathbf{e}) = \mathbf{e} + \sum_{k=1}^{\lfloor z/2 \rfloor} \mathbf{\Lambda}^{2k}\mathbf{e} - \sum_{k=1}^{\lceil z/2 \rceil} \mathbf{\Lambda}^{2k-1}\mathbf{e},$$

as the *alternate outcoming weighted Bonacich centrality* measure applied to $\mathbf{e}$, and the vector

$$\mathbf{b}^{-}_{\text{alt}}(\mathbf{\Lambda}, \mathbf{e}) = (\mathbf{I} + \mathbf{\Lambda}^T)^{-1}\mathbf{e} = \mathbf{e} + \sum_{k=1}^{z} (-1)^k (\mathbf{\Lambda}^T)^k \mathbf{e},$$

that is,

$$\mathbf{b}^{-}_{\text{alt}}(\mathbf{\Lambda}, \mathbf{e}) = \mathbf{e} + \sum_{k=1}^{\lfloor z/2 \rfloor} (\mathbf{\Lambda}^T)^{2k}\mathbf{e} - \sum_{k=1}^{\lceil z/2 \rceil} (\mathbf{\Lambda}^T)^{2k-1}\mathbf{e},$$

as the *alternate incoming weighted Bonacich centrality* measure[4] applied to $\mathbf{e}$. It follows that the equilibrium profile of game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ may be expressed in terms of the alternate incoming weighted Bonacich centrality vector applied to the vector of equilibrium peaks, i.e.,

$$\hat{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}}(\mathbf{\Omega}, \mathbf{a}^*).$$

Remark 3. If for some i, $a^*_i - \sum_{j:j \neq i} \omega_{ji}\hat{a}_j \leq 0$, these players are "inactive" at equilibrium, i.e., $\hat{a}_i = 0$. Let $L = \{l : \hat{a}_l > 0\}$ be the set of "active" players. Theorem 2 may be applied to the subgame $G(\mathbf{\Omega}_L, \mathbf{a^*}_L)$ where $\mathbf{\Omega}_L$ and $\mathbf{a^*}_L$

[4]Note that $\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Lambda}, \mathbf{e}) = \mathbf{b}^{+}(-\mathbf{\Lambda}, \mathbf{e})$ and $\mathbf{b}^{-}_{\text{alt}}(\mathbf{\Lambda}, \mathbf{e}) = \mathbf{b}^{-}(-\mathbf{\Lambda}, \mathbf{e})$.

denote, respectively, the new network and the new vector of equilibrium peaks obtained after deletion of all the inactive players.

4 The efficient allocation

To characterize the efficient allocation of water extraction activities, we take a standard utilitarian approach. Given the structure and intensities of hydrological influences $\mathbf{\Omega}$, the maximum social welfare, W, can be determined by solving,

$$W(\mathbf{\Omega}, \mathbf{a}) = \text{Max} \sum_i [p_i a_i - q_i(a_i + \sum_{j:j\neq i} \omega_{ji} a_j)].$$

We say a profile is *efficient* for a given network of sources if, and only if, there is no other profile that leads to a strictly higher social welfare. Given a network of source $\mathbf{\Omega}$, the efficient profile[5] $\tilde{\mathbf{a}} \in (\mathbb{R}_+)^N$ satisfies, for all i,

$$\tilde{a}_i > 0 \iff p_i - q_i'(\sum_{j:j\neq i} \omega_{ji}\tilde{a}_j) > \sum_{k:k\neq i} \omega_{ik} q_k'(\tilde{a}_k + \sum_{l:l\neq k,i} \omega_{lk}\tilde{a}_l).$$

Then, the first order conditions for efficiency are

$$\begin{cases} \text{if } \tilde{a}_i > 0, & \text{then } p_i - q_i'(\tilde{a}_i + \sum\limits_{j:j\neq i} \omega_{ji}\tilde{a}_j) = \sum\limits_{k:k\neq i} \omega_{ik} q_k'(\tilde{a}_k + \sum\limits_{l:l\neq k} \omega_{lk}\tilde{a}_l). \\ \text{if } \tilde{a}_i = 0, & \text{then } p_i - q_i'(\tilde{a}_i + \sum\limits_{j:j\neq i} \omega_{ji}\tilde{a}_j) \leq \sum\limits_{k:k\neq i} \omega_{ik} q_k'(\tilde{a}_k + \sum\limits_{l:l\neq k} \omega_{lk}\tilde{a}_l). \end{cases}$$

If all communities have a positive extraction activity, i.e., for all i, $\tilde{a}_i > 0$, the efficient extraction activity of a community may be expressed in terms of its centrality in the original network. The proof is in the Appendix and

[5]The efficient profile is unique because the maximum social welfare W is a strictly concave function.

the intuition is as follows.

Since every directed acyclic graph has at least one vertex with no out-coming arcs, every network game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ contains at least one community with no successor. We refer to these participants as the *sink players of rank 1* and their set is noted D_1. For all $i \in D_1$, the efficient extraction activity verifies

$$p_i - q_i'(\tilde{a}_i + \sum_{j:j\neq i} \omega_{ji}\tilde{a}_j) = 0,$$

that is,

$$\tilde{a}_i + \sum_{j:j\neq i} \omega_{ji}\tilde{a}_j = (q_i')^{-1}(p_i).$$

Next, consider the subgame obtained after deletion of all the sink players of rank 1 and their in-coming flows. Since any subgraph of an acyclic graph is also acyclic, a new set of sink players appears. Let call them the *sink players of rank 2.* Their set is denoted D_2. For all $k \in D_2$, the efficient extraction activity verifies

$$p_k - q_k'(\tilde{a}_k + \sum_{l:l\neq k} \omega_{lk}\tilde{a}_l) = \sum_{i\in D_1} \omega_{ki} q_i'(\tilde{a}_i + \sum_{j:j\neq i} \omega_{ji}\tilde{a}_j) = \sum_{i\in D_1} \omega_{ki} p_i.$$

that is,

$$\tilde{a}_k + \sum_{l:l\neq k} \omega_{lk}\tilde{a}_l = (q_k')^{-1}(p_k - \sum_{i\in D_1} \omega_{ki} p_i) = (q_k')^{-1}(\tilde{p}_k),$$

where $\tilde{p}_k = p_k - \sum_{i\in D_1} \omega_{ki} p_i$ denote player k's *efficient marginal benefit.*

Then, repeat the same procedure until the last set of sink players is considered. If the length of the longest directed path in G is p, the procedure is repeated $p + 1$ times. We show in the Appendix that, for all i,

$$\tilde{a}_i + \sum_{j:j\neq i} \omega_{ji}\tilde{a}_j = (q_i')^{-1}(p_i - \sum_{k:k\neq i} \omega_{ik}\tilde{p}_k) = (q_i')^{-1}(\tilde{p}_i).$$

Therefore, the N-vector of efficient marginal benefits $\tilde{\mathbf{p}}$ may be expressed as a function of both $\mathbf{\Omega}$ and $\mathbf{p}$.

Let $\tilde{a}_i^* = (q_i')^{-1}(\tilde{p}_i)$ denote player i's *efficient peak*. Since for all i, $\tilde{a}_i > 0$, it appears that,

$$\tilde{a}_i = \tilde{a}_i^* - \sum_{j:j\neq i} \omega_{ji}\tilde{a}_j,$$

which results in a system of N linear equations that may be solved by applying the induction reasoning used for the proof of Theorem 2. Consider the sets of source players $C_1, C_2, \ldots, C_{p+1}$ defined in the previous section. For all $i \in C_1$,

$$\tilde{a}_i = \tilde{a}_i^*,$$

for all $j \in C_2$,

$$\hat{a}_j = \tilde{a}_j^* - \sum_{i\in C_1} \omega_{ij}\tilde{a}_i = \tilde{a}_j^1,$$

and so on until the last set of source players C_{p+1} whose efficient extraction activity is equal to their p-times modified efficient peak. This leads to the following result.

Theorem 3. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ be a network game and p the length of the longest directed path in $\mathbf{\Omega}$. If for all i, $\tilde{a}_i^* - \sum_{j:j\neq i}\omega_{ji}\tilde{a}_j > 0$, the efficient profile is*

$$\tilde{\mathbf{a}} = \tilde{\mathbf{a}}^* + \sum_{k=1}^{\lfloor p/2 \rfloor} (\mathbf{\Omega}^T)^{2k}\tilde{\mathbf{a}}^* - \sum_{k=1}^{\lceil p/2 \rceil} (\mathbf{\Omega}^T)^{2k-1}\tilde{\mathbf{a}}^*,$$

where

$$\tilde{\mathbf{a}}^* = (Q')^{-1}(\mathbf{p} + \sum_{k=1}^{\lfloor p/2 \rfloor} \mathbf{\Omega}^{2k}\mathbf{p} - \sum_{k=1}^{\lceil p/2 \rceil} \mathbf{\Omega}^{2k-1}\mathbf{p}).$$

The equilibrium effect described in the previous section is still present at efficiency. However, this effect is now counterbalanced by another network effect contained in $\tilde{\mathbf{a}}^*$. The first summation in the expression of $\tilde{\mathbf{a}}^*$ counts the total weight of even directed directed paths that start from the corresponding vertex in G, and the second summation counts the total weight of odd directed paths that start from it, where directed paths that end at j in G are weighted by p_j.

Since for all i, $q_i(.)$ is strictly convex, the first sum in $\tilde{\mathbf{a}}^*$ tells that the efficient extraction activity of a community is positively related with the weight of even length directed paths that start from it. In contrast, the negative sign on the second summation means the efficient extraction activity of a community is negatively related with the weight of odd length directed paths that start from it. These outcoming "substitutability-complementarity" effects are given by

$$\mathbf{E}^{\text{ef}} = \sum_{k=1}^{\lfloor p/2 \rfloor} \mathbf{\Omega}^{2k} - \sum_{k=1}^{\lceil p/2 \rceil} \mathbf{\Omega}^{2k-1},$$

which may be called the *efficiency effects* matrix. Moreover, these effects are weighted by the marginal costs. For a community, the higher the marginal costs, the lower the impact of the efficiency effects.

To fix ideas on these results, we study the following example.

Example 1 (Sharing water in a river basin)**.** Consider an hydrological unit composed of five sources of water s_i, $i = 1, \ldots, 5$. There are three "rivers". The main one starts from s_1 and ends at s_3. The two other ones are tributaries of the main river. One starts from s_4 and ends at s_2, and the other one starts from s_5 and ends at s_3. The sources s_2 and s_3 are both at the confluence of the main river with one tributary.

There are five communities and each community extracts water from its own single source, i.e., the community i extracts water from the source s_i. We assume that communities are "price-taker", so the marginal benefits are constant and uniform. For all community i, let $p_i = 1$ and a_i denotes the

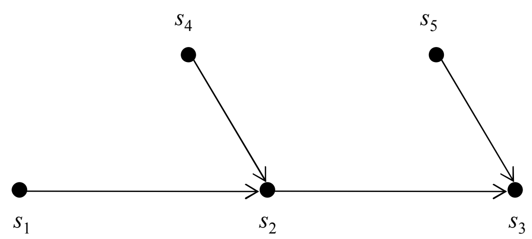

Figure 2: An hydrological unit with five water sources

water extraction activity. For all community without predecessors, i.e., for $i = 1, 4, 5$, the utility function U_i is given by

$$U_i = a_i - (a_i)^2.$$

We suppose that hydrological influences are more important along the tributaries than along the main river. Moreover, we assume that communities being at the confluence of two rivers have lower marginal costs of water extraction than those whose water source is feed by only one river. For the community that extracts water from s_2, i.e., for $i = 2$, the utility function U_2 is given by

$$U_2 = a_2 - \frac{3}{4}(a_2 + \frac{1}{5}a_1 + \frac{1}{3}a_4)^2,$$

where $(1/5)a_1$ denotes the impact from its predecessor along the main river and $(1/3)a_4$ the impact from its predecessor along the tributary. For the community that extracts water from s_3, i.e., for $i = 3$, the utility function U_3 is given by

$$U_3 = a_3 - \frac{3}{4}(a_3 + \frac{1}{5}a_2 + \frac{1}{3}a_5)^2,$$

where $(1/5)a_2$ denotes the impact from its predecessor along the main river and $(1/3)a_5$ the impact from its predecessor along the tributary.

The weighted adjacency matrix $\mathbf{\Omega}$ of the graph that represents the network of sources is

$$\mathbf{\Omega} = \begin{pmatrix} 0 & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}.$$

The length of the longest directed path in the graph is $p = 2$. It follows that the equilibrium effects matrix is given by

$$\mathbf{E}^{\text{eq}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{5} & 0 & 0 & -\frac{1}{3} & 0 \\ +\frac{1}{25} & -\frac{1}{5} & 0 & +\frac{1}{15} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix summarizes the intensity of the hydrological influences between the sources. In particular, it shows that the community that extracts water from s_3 is positively impacted by both the community that extracts water from s_1 and the community that extracts water from s_4. This is due to the fact these communities are two-links-away from each other in the network of sources and therefore, their extraction activities are strategic complements.

Since for $i = 1, 4, 5$, $a_i^* = 1/2$, and for $i = 2, 3$, $a_i^* = 2/3$, the Nash equilibrium profile of the game played by the communities is

$$\hat{\mathbf{a}} = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} - \frac{1}{5} \times \frac{1}{2} - \frac{1}{3} \times \frac{1}{2} \\ \frac{2}{3} + \frac{1}{25} \times \frac{1}{2} - \frac{1}{5} \times \frac{2}{3} + \frac{1}{15} \times \frac{1}{2} - \frac{1}{3} \times \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.50 \\ 0.40 \\ 0.42 \\ 0.50 \\ 0.50 \end{pmatrix}.$$

At equilibrium, $U_1 = U_4 = U_5 = 0.25$, $U_2 \approx 0.07$ and $U_3 \approx 0.09$. Clearly, the most disadvantaged communities, in terms of both water extraction activity and utility level, are the ones that extract water from s_2 and s_3.

To obtain the efficient profile[6], we compute the efficiency effects matrix,

$$\mathbf{E}^{\text{ef}} = \begin{pmatrix} 0 & -\frac{1}{5} & +\frac{1}{25} & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & +\frac{1}{15} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \end{pmatrix}.$$

Since for $i = 1, 4, 5$, $(q_i')^{-1}(x) = (1/2)x$, and for $i = 2, 3$, $(q_i')^{-1}(x) = (2/3)x$, the vector of efficient peaks is

$$\tilde{\mathbf{a}}^* = \begin{pmatrix} (1 - \frac{1}{5} + \frac{1}{25}) \times \frac{1}{2} \\ (1 - \frac{1}{5}) \times \frac{2}{3} \\ 1 \times \frac{2}{3} \\ (1 - \frac{1}{3} + \frac{1}{15}) \times \frac{1}{2} \\ (1 - \frac{1}{3}) \times \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{21}{50} \\ \frac{8}{15} \\ \frac{2}{3} \\ \frac{11}{30} \\ \frac{1}{3} \end{pmatrix}.$$

[6]Note that in the example, the sets of source players and the sets of sink players are not symmetric, i.e., $C_3 = D_1 = \{3\}$ but $C_2 = \{2\} \neq D_2 = \{2, 5\}$ and $C_1 = \{1, 4, 5\} \neq D_3 = \{1, 4\}$.

and therefore, the efficient profile of the game played by the communities is

$$\hat{\mathbf{a}} = \begin{pmatrix} \frac{21}{50} \\ \frac{8}{15} - \frac{1}{5} \times \frac{21}{50} - \frac{1}{3} \times \frac{11}{30} \\ \frac{2}{3} + \frac{1}{25} \times \frac{21}{50} - \frac{1}{5} \times \frac{8}{15} + \frac{1}{15} \times \frac{11}{30} - \frac{1}{3} \times \frac{1}{3} \\ \frac{11}{30} \\ \frac{1}{3} \end{pmatrix} \approx \begin{pmatrix} 0.42 \\ 0.33 \\ 0.49 \\ 0.37 \\ 0.33 \end{pmatrix}.$$

At efficiency, disparities of utility levels decrease since $U_1 \approx 0.24$, $U_2 \approx 0.12$, $U_3 \approx 0.16$, $U_4 \approx 0.23$ and $U_5 \approx 0.22$.

At equilibrium, water is over-consumed. However, not all communities over-consume water. Indeed, there is only one community whose equilibrium extraction activity is less than its efficient level: the one at the downstream tail of the main river. The community located at the downstream tail of a tributary but with successors along the main river, i.e., the community that extracts water from s_2, has a lower efficient activity than its equilibrium level. However, its utility level is greater at efficiency, because all of its predecessors have a lower extraction activity.

In this example, to achieve the efficient allocation of water, the communities without predecessors should decrease their equilibrium extraction activity and renounce to some benefits. Then, the other communities would get more benefits. Moreover, the community without successors should be the only one to increase its equilibrium extraction activity.

To conclude this section, it should be noted that assuming a positive efficient profile implies a positive vector of efficient marginal benefits, and that Theorem 3 characterizes the efficient profile in terms of the centrality of a community in the network of sources and is valid for all kind of efficient profile.

Remark 4. For all i,

$$\tilde{a}_i = (q_i')^{-1}(\tilde{p}_i) - \sum_{j:j\neq i} \omega_{ji}\tilde{a}_j > 0 \Longrightarrow \tilde{p}_i = p_i - \sum_{j:j\neq i} \omega_{ij}\tilde{p}_j > 0.$$

Remark 5. Following the definitions of the alternate weighted Bonacich centrality measures (incoming and outcoming) provided at Remark 2, we may express the efficient profile in terms of the alternate incoming weighted Bonacich centrality vector applied to the vector of efficient peaks, which may itself be expressed in terms of the alternate outcoming weighted Bonacich centrality vector applied to the vector of marginal benefits, i.e.,

$$\tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}}(\boldsymbol{\Omega}, \tilde{\mathbf{a}}^*) = \mathbf{b}^{-}_{\text{alt}}(\boldsymbol{\Omega}, (Q')^{-1}[\mathbf{b}^{+}_{\text{alt}}(\boldsymbol{\Omega}, \mathbf{p})]).$$

Remark 6. If for some i, $(q_i')^{-1}(\tilde{p}_i) - \sum_{j:j\neq i} \omega_{ji}\tilde{a}_j \leq 0$, these players are "inactive" at efficiency, i.e., $\tilde{a}_i = 0$. The inactivity of a player may come from two sources: one resulting from the calculation of the efficient peaks, and the other resulting from the calculation of the efficient extraction activities. Let $S = \{s : \tilde{a}^*_s > 0\}$ be the set of players with positive efficient peaks and $T = \{t : \tilde{a}_t > 0\}$ the set of "active" players. The vector of efficient peaks may be calculated from the new network $\boldsymbol{\Omega}_S$ obtained after deletion of all the players with non-positive efficient marginal benefits. Let $\tilde{\mathbf{a}}^S$ be the resulting vector of efficient peaks. Theorem 3 may then be applied to the subgame $G(\boldsymbol{\Omega}_T, \tilde{\mathbf{a}}^S{}_T)$ where $\boldsymbol{\Omega}_T$ and $\tilde{\mathbf{a}}^S{}_T$ denote, respectively, the new network and the new vector of efficient peaks obtained after deletion of all the inactive players.

5 Discussion

The optimal tax plan to achieve the efficient allocation of water involves different tax rates at each source of water throughout the network of sources. Following Theorem 3, if for all i, $\tilde{a}_i > 0$, the optimal tax plan of game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ is

$$\boldsymbol{\tau} = [\sum_{k=1}^{\lceil p/2 \rceil} \mathbf{\Omega}^{2k-1} - \sum_{k=1}^{\lfloor p/2 \rfloor} \mathbf{\Omega}^{2k}]\mathbf{p},$$

or equivalently,

$$\boldsymbol{\tau} = -\mathbf{E}^{\text{ef}}\mathbf{p}.$$

Therefore, the optimal tax rate of a community is positively related with the weight of odd length directed paths that start at it, and negatively related with the weight of even length directed paths that start at it, where directed paths that end at j in G are weighted by p_j. The optimal tax plan is the opposite of the efficiency effect matrix, weighted by the vector of marginal benefits. For a community, the highest its efficiency effects, the lowest its optimal tax rate.

Suppose there is a social planner who knows the game $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ perfectly, and who wants to encourage communities to play the efficient extraction activity profile without decreasing the utility levels. The social planner may implement the optimal tax plan $\boldsymbol{\tau}$ and then, redistribute the tax revenue via direct transfer payments. The implementation of the optimal tax policy will encourage communities to play the efficient profile, and the implementation of lump sum subsidies will keep the utilities efficient. The strategy of utility-maximizing water extraction activities involves a new simultaneous-move game.

Let τ_i be the i-entry of the tax vector. The maximum utility may be determined by solving, for all i,

$$\breve{U}_i(\mathbf{\Omega}, \mathbf{a}) = \text{Max } p_i a_i - q_i(a_i + \sum_{j:j\neq i} \omega_{ji} a_j) + \tau_i(\tilde{a}_i - a_i)$$

where $\tilde{a}_i$ is given, since it is known by the social planner. It follows that a water extraction profile $\breve{\mathbf{a}} \in (\mathbb{R}_+)^N$ is a Nash equilibrium of the new game played by the communities if, and only if, for all i,

$$\breve{a}_i = \max\{0, \breve{a}_i^* - \sum_{j:j\neq i} \omega_{ji} \breve{a}_j\}.$$

where $\breve{a}_i^* = (q_i')^{-1}(p_i - \tau_i)$. Following Theorem 2, if for all i, $\breve{a}_i^* - \sum\limits_{j:j\neq i} \omega_{ji} \breve{a}_j \geq 0$, the Nash equilibrium profile is

$$\breve{\mathbf{a}} = [\mathbf{I} + \sum_{k=1}^{\lfloor p/2 \rfloor} (\mathbf{\Omega}^T)^{2k} - \sum_{k=1}^{\lceil p/2 \rceil} (\mathbf{\Omega}^T)^{2k-1}](Q')^{-1}(\mathbf{p} - \boldsymbol{\tau}),$$

where p is the length of the longest directed path in $\mathbf{\Omega}$. It follows that

$$\breve{\mathbf{a}} = [\mathbf{I} + \sum_{k=1}^{\lfloor p/2 \rfloor} (\mathbf{\Omega}^T)^{2k} - \sum_{k=1}^{\lceil p/2 \rceil} (\mathbf{\Omega}^T)^{2k-1}](Q')^{-1}([\mathbf{I} + \sum_{k=1}^{\lfloor p/2 \rfloor} \mathbf{\Omega}^{2k} - \sum_{k=1}^{\lceil p/2 \rceil} \mathbf{\Omega}^{2k-1}]\mathbf{p}),$$

and therefore, $\breve{\mathbf{a}} = \tilde{\mathbf{a}}$.

The optimal tax plan reflects the marginal damages a community inflicts on other communities at the efficient extraction activity profile. In that sense, the optimal tax plan may be seen as a Pigouvian tax plan. However, since the network game played by the communities also involve complementarities for players who are two-links-away from each other in the network of sources, the optimal tax plan also reflects the marginal benefits a community brings to other communities.

The efficient utility levels are achieved thanks to the redistribution of the revenue generated by the optimal tax plan. That is, for all i,

$$\begin{aligned}
\breve{U}_i(\boldsymbol{\Omega}, \breve{\mathbf{a}}) &= p_i \breve{a}_i - q_i(\breve{a}_i + \textstyle\sum_{j:j\neq i} \omega_{ji} \breve{a}_j) + \tau_i(\tilde{a}_i - \breve{a}_i) \\
&= p_i \tilde{a}_i - q_i(\tilde{a}_i + \textstyle\sum_{j:j\neq i} \omega_{ji} \tilde{a}_j) + \tau_i(\tilde{a}_i - \tilde{a}_i) \\
&= p_i \tilde{a}_i - q_i(\tilde{a}_i + \textstyle\sum_{j:j\neq i} \omega_{ji} \tilde{a}_j) \\
&= U_i(\boldsymbol{\Omega}, \tilde{\mathbf{a}}).
\end{aligned}$$

To conclude this paper, we illustrate these results by computing the optimal tax plan and the vector of lump sum subsidies for the game studied at Example 1.

Example 2 (Example 1 continued)**.** The i-entry of the vector of lump sum subsidies $\mathbf{s}$ is the product between the optimal tax rate τ_i and the efficient extraction activity $\tilde{a}_i$. Since for all i, $p_i = 1$, we obtain

$$\boldsymbol{\tau} \approx \begin{pmatrix} 0.16 \\ 0.20 \\ 0.00 \\ 0.27 \\ 0.33 \end{pmatrix} \quad \text{and} \quad \mathbf{s} \approx \begin{pmatrix} 0.07 \\ 0.07 \\ 0.00 \\ 0.10 \\ 0.11 \end{pmatrix}$$

The highest tax rates are imposed to the communities located at the upstream tail of the tributaries, because this communities deliver important marginal damages and few marginal benefits to other communities. For instance, the tax rate of the community that extracts water from s_5 only reflects the marginal damage inflicted on the community that exploits s_3. The highest transfer payments are distributed to the communities without predecessors because these communities do not benefit from the changes in water extraction activity by other communities. Their subsidy only reflects the decrease of their utility level through the decrease of their own water extraction activity. Finally, since the community that exploits s_3 has no successors, this community is not taxed and therefore not subsidized.

These results create space for further research. We list three useful directions. Firstly, the intuition that our findings also apply to cyclic networks, provided that the spectral radius of the weighted adjacency matrix is sufficiently low, needs to be formally studied. Secondly, the optimal tax plan designed in this paper raises issues as to the how water rights might be defined. Thirdly, several refinements of the model could be undertaken. For instance, the availability of water resources could be constrained, at the individual level and/or at the system level. Another extension refers to the fact that participants could exploit several water sources, and share some of them with other participants.

6 Appendix

6.1 Proof of Theorem 1

Algorithm Nash equilibrium in wDAG (**Data:** $\mathbf{\Omega}$, $\mathbf{a}^*$; **Result:** $\hat{\mathbf{a}}$)

```
S ← {i : Ω_{.i} = 0}
T ← 𝒩 \ S
while S ≠ ∅ do
    for all i ∈ S do
        â_i ← a*_i
        for all j ∈ T do
            a*_j ← a*_j − ω_{ij} â_i
            if a*_j < 0 then                ▷ No negative activity
                a*_j ← 0
            end if
        end for
    end for
    Ω ← Ω_{T×T}                             ▷ The graph is reduced
    a* ← a*_T
    S ← {i : Ω_{.i} = 0}
    T ← T \ S
end while
```

□

6.2 Lemma A1

Definition A1. Let $\mathbf{X}$ be a square matrix. Then $\mathbf{X}$ is *nilpotent* if $\mathbf{X}^k = 0$ for some positive integer k.

Lemma A1. *Let G be a weighted graph, $\mathbf{\Omega}$ its weighted adjacency matrix and $\mathbf{M}$ its unweighted adjacency matrix. Then, $\mathbf{\Omega}$ is nilpotent if and only if $\mathbf{M}$ is nilpotent.*

Proof. Let G be a weighted graph, $\mathbf{\Omega}$ its weighted adjacency matrix and $\mathbf{M}$ its unweighted adjacency matrix. Denote by $\overline{\omega}, \underline{\omega}$, the greatest and smallest positive entry of $\mathbf{\Omega}$,

$$\overline{\omega} = \max_{kl} \omega_{kl} \quad \text{and} \quad \underline{\omega} = \min_{kl}\{\omega_{kl} : \omega_{kl} > 0\}.$$

Let us prove by induction, that $\forall n \in \mathbb{N}$,

$$\underline{\omega}^n \mathbf{M}^n \leq \mathbf{\Omega}^n \leq \overline{\omega}^n \mathbf{M}^n.$$

For $n = 1$, it is immediate. Assume it holds for some $n \geq 1$. Then, for all kl,

$$\begin{aligned}
\omega_{kl}^{(n+1)} &= \textstyle\sum_p \omega_{kp}^{(n)} \omega_{pl} \\
&\leq \textstyle\sum_p \overline{\omega}^n m_{kp}^{(n)} \overline{\omega} m_{pl}, \text{ by induction hypothesis} \\
&= \overline{\omega}^{n+1} \textstyle\sum_p m_{kp}^{(n)} m_{pl} \\
&= \overline{\omega}^{n+1} m_{kl}^{(n+1)}
\end{aligned}$$

where $\omega_{kl}^{(r)}, m_{kl}^{(r)}$ denote the kl-entries in $\mathbf{\Omega}^r, \mathbf{M}^r$ for $r = n, n+1$. Hence, $\mathbf{\Omega}^{n+1} \leq \overline{\omega}^{n+1}\mathbf{M}^{n+1}$ holds. The proof is similar for the other inequality. Therefore, by induction the inequalities have been established. It follows that $\mathbf{\Omega}$ and $\mathbf{M}$ are simultaneously nilpotent and are moreover nilpotent of same order. $\square$

6.3 Proof of Theorem 2

Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ be a network game with $\mathbf{\Omega} \geq 0$ the weighted adjacency matrix of a wDAG and $\mathbf{a}^* >> 0$ a vector of equilibrium peaks. Let p denote the length of the longest directed path in the wDAG.

If for all i, $a_i^* - \sum_{j:j\neq i} \omega_{ji}\hat{a}_j \geq 0$, let us show by induction that, for all i,

$$\hat{a}_i = (\mathbf{a}^* + \sum_{k=1}^{\lfloor p/2 \rfloor} (\mathbf{\Omega}^T)^{2k}\mathbf{a}^* - \sum_{k=1}^{\lceil p/2 \rceil} (\mathbf{\Omega}^T)^{2k-1}\mathbf{a}^*)_i,$$

or equivalently,

$$\hat{a}_i = ([\sum_{k=0}^{p} (-1)^k (\mathbf{\Omega}^T)^k]\mathbf{a}^*)_i.$$

Since the graph is acyclic, there exists at least one player with no predecessors. Let C_1 denote the set of players who have no predecessors (call them the *source players of rank 1*). Let C_x denote the set of players who have no predecessors in the subgraph where all source players of rank $r < x$, and their outcoming links, have been deleted (call them the *source players of rank* x). The maximum value integer x can have is $p + 1$.

For all $i \in C_1$,

$$\hat{a}_i = a_i^* = ((-1)^0 (\mathbf{\Omega}^T)^0 \mathbf{a}^*)_i.$$

The ij-entry of $(\mathbf{\Omega}^T)^k$ denote the total weight of all directed paths with length k starting at vertex j and ending at vertex i. By Lemma 8.1.2 in Godsil and Royle (2001, p. 165), and since the graph is acyclic, $((\mathbf{\Omega}^T)^k)_{ij} = 0$ for all $i \in C_1$, for all j and for all integer $k \geq 1$. Therefore, the property holds.

Let $x \leq p$. Assume the property holds for $C_{1,x} = C_1 \cup C_2 \cup \ldots \cup C_x$. That is, for all $l \in C_{1,x}$,

$$\hat{a}_l = ([\sum_{k=0}^{p} (-1)^k (\mathbf{\Omega}^T)^k]\mathbf{a}^*)_l.$$

For all $m \in C_{x+1}$,

$$\hat{a}_m = a_m^* - \sum_{l:l\neq m} \omega_{lm}\hat{a}_l = a_m^* - \sum_{l \in C_{1,x}} \omega_{lm}\hat{a}_l$$

By induction hypothesis, it follows that

$$\begin{aligned}\hat{a}_m &= ((-1)^0(\mathbf{\Omega}^T)^0\mathbf{a}^*)_m - \textstyle\sum_{l\in C_{1,x}}\omega_{lm}([\sum_{k=0}^{p}(-1)^k(\mathbf{\Omega}^T)^k]\mathbf{a}^*)_l \\ &= ((-1)^0(\mathbf{\Omega}^T)^0\mathbf{a}^*)_m - (\mathbf{\Omega}^T)_{ml}([\textstyle\sum_{k=0}^{p}(-1)^k(\mathbf{\Omega}^T)^k]\mathbf{a}^*)_l \\ &= ((-1)^0(\mathbf{\Omega}^T)^0\mathbf{a}^*)_m - ((\mathbf{\Omega}^T)[\textstyle\sum_{k=0}^{p}(-1)^k(\mathbf{\Omega}^T)^k]\mathbf{a}^*)_m\end{aligned}$$

since $(\mathbf{\Omega}^T)_{ml} = 0$ for all $l \notin C_{1,x}$. Then,

$$\begin{aligned}\hat{a}_m &= ((-1)^0(\mathbf{\Omega}^T)^0\mathbf{a}^*)_m + ([\textstyle\sum_{k=1}^{p+1}(-1)^k(\mathbf{\Omega}^T)^k]\mathbf{a}^*)_m \\ &= ([\textstyle\sum_{k=0}^{p+1}(-1)^k(\mathbf{\Omega}^T)^k]\mathbf{a}^*)_m \\ &= ([\textstyle\sum_{k=0}^{p}(-1)^k(\mathbf{\Omega}^T)^k]\mathbf{a}^*)_m\end{aligned}$$

since there is no directed path of length equal or greater than $p+1$ ending at vertex m, i.e., $((\mathbf{\Omega}^T)^{p+1})_m = \mathbf{0}$ for all $m \in C_{x+1}$. Therefore, the property is true for $x+1$. So by induction, it will be true for $x = p+1$. □

6.4 Proof of Theorem 3

Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{a}^*)$ be a network game with $\mathbf{\Omega} \geq 0$ the weighted adjacency matrix of a wDAG and $\mathbf{a}^* >> 0$ a vector of equilibrium peaks. Let p denote the length of the longest directed path in the wDAG.

If for all i, $\tilde{a}_i^* - \sum_{j:j\neq i}\omega_{ji}\tilde{a}_j > 0$, let us show by two inductions that, for all i,

$$\tilde{a}_i = (\tilde{\mathbf{a}}^* + \sum_{k=1}^{\lfloor p/2\rfloor}(\mathbf{\Omega}^T)^{2k}\tilde{\mathbf{a}}^* - \sum_{k=1}^{\lceil p/2\rceil}(\mathbf{\Omega}^T)^{2k-1}\tilde{\mathbf{a}}^*)_i,$$

where

$$\tilde{\mathbf{a}}^* = (Q')^{-1}(\mathbf{p} + \sum_{k=1}^{\lfloor p/2\rfloor}\mathbf{\Omega}^{2k}\mathbf{p} - \sum_{k=1}^{\lceil p/2\rceil}\mathbf{\Omega}^{2k-1}\mathbf{p}),$$

or equivalently,

$$\tilde{a}_i = ([\sum_{k=0}^{p}(-1)^k(\mathbf{\Omega}^T)^k](Q')^{-1}([\sum_{k=0}^{p}(-1)^k\mathbf{\Omega}^k]\mathbf{p}))_i.$$

First induction. Let us show that, for all i,

$$\tilde{a}_i = (q_i')^{-1}(([\sum_{k=0}^{p}(-1)^k\mathbf{\Omega}^k]\mathbf{p})_i) - \sum_{j:j\neq i}\omega_{ji}\tilde{a}_j.$$

Since the graph is acyclic, there exists at least one player with no successors. Let D_1 denote the set of players who have no successors (call them the *sink players of rank 1*). Let D_x denote the set of players who have no successors in the subgraph where all sink players of rank $r < x$, and their incoming links, have been deleted (call them the *sink players of rank* x). The maximum value integer x can have is $p+1$.

For all $i \in D_1$,

$$\begin{aligned}\tilde{a}_i &= (q_i')^{-1}(p_i) - \sum_{j:j\neq i}\omega_{ji}\tilde{a}_j \\ &= (q_i')^{-1}(((-1)^0\mathbf{\Omega}^0\mathbf{p})_i) - \sum_{j:j\neq i}\omega_{ji}\tilde{a}_j\end{aligned}$$

The ij-entry of $\mathbf{\Omega}^k$ denote the total weight of all directed paths with length k starting at vertex i and ending at vertex j. By Lemma 8.1.2 in Godsil and Royle (2001, p. 165), and since the graph is acyclic, $(\mathbf{\Omega}^k)_{ij} = 0$ for all $i \in D_1$, for all j and for all integer $k \geq 1$. Therefore, the property holds.

Let $x \leq p$. Assume the property holds for $D_{1,x} = D_1 \cup D_2 \cup \ldots D_x$. That is, for all $l \in D_{1,x}$,

$$\tilde{a}_l = (q_l')^{-1}(([\sum\nolimits_{k=0}^{p}(-1)^k\mathbf{\Omega}^k]\mathbf{p})_l) - \sum\nolimits_{j:j\neq l}\omega_{jl}\tilde{a}_j$$

For all $m \in D_{x+1}$,

$$\begin{aligned}\tilde{a}_m &= (q'_m)^{-1}(p_m - \textstyle\sum_{l:l\neq m} \omega_{ml} q'_l(\tilde{a}_l + \sum_{j:j\neq l} \omega_{jl}\tilde{a}_j)) - \sum_{n:n\neq m} \omega_{nm}\tilde{a}_n \\ &= (q'_m)^{-1}(p_m - \textstyle\sum_{l\in D_{1,x}} \omega_{ml} q'_l(\tilde{a}_l + \sum_{j:j\neq l} \omega_{jl}\tilde{a}_j))\end{aligned}$$

since $(\mathbf{\Omega})_{nm} = 0$ for all n. By induction hypothesis, it follows that

$$\begin{aligned}\tilde{a}_m &= (q'_m)^{-1}(((-1)^0\mathbf{\Omega}^0\mathbf{p})_m - \textstyle\sum_{l\in D_{1,x}} \omega_{ml}([\sum_{k=0}^{p} (-1)^k\mathbf{\Omega}^k]\mathbf{p})_l) \\ &= (q'_m)^{-1}(((-1)^0\mathbf{\Omega}^0\mathbf{p})_m - \mathbf{\Omega}_{ml}([\textstyle\sum_{k=0}^{p} (-1)^k\mathbf{\Omega}^k]\mathbf{p})_l) \\ &= (q'_m)^{-1}(((-1)^0\mathbf{\Omega}^0\mathbf{p})_m - (\mathbf{\Omega}[\textstyle\sum_{k=0}^{p} (-1)^k\mathbf{\Omega}^k]\mathbf{p})_m)\end{aligned}$$

since $(\mathbf{\Omega})_{ml} = 0$ for all $l \notin D_{1,x}$. Then,

$$\begin{aligned}\tilde{a}_m &= (q'_m)^{-1}(((-1)^0\mathbf{\Omega}^0\mathbf{p})_m + ([\textstyle\sum_{k=1}^{p+1} (-1)^k\mathbf{\Omega}^k]\mathbf{p})_m) \\ &= (q'_m)^{-1}(([\textstyle\sum_{k=0}^{p+1} (-1)^k\mathbf{\Omega}^k]\mathbf{p})_m) \\ &= (q'_m)^{-1}(([\textstyle\sum_{k=0}^{p} (-1)^k\mathbf{\Omega}^k]\mathbf{p})_m)\end{aligned}$$

since there is no directed path of length equal or greater than $p+1$ starting at vertex m, i.e., $(\mathbf{\Omega}^{p+1})_m = \mathbf{0}$ for all $m \in D_{x+1}$. Therefore, the property is true for $x+1$. So by induction, it will be true for $x = p+1$.

Second induction. To show that, for all i,

$$\tilde{a}_i = ([\sum_{k=0}^{p} (-1)^k(\mathbf{\Omega}^T)^k](Q')^{-1}([\sum_{k=0}^{p} (-1)^k\mathbf{\Omega}^k]\mathbf{p})_i,$$

apply Theorem 2 to the following system of N equations,

$$\text{for all } i, \quad \tilde{a}_i = (q_i')^{-1}(([\sum_{k=0}^{p}(-1)^k \mathbf{\Omega}^k]\mathbf{p})_i) - \sum_{j:j\neq i}\omega_{ji}\tilde{a}_j,$$

with $\hat{a}_i = \tilde{a}_i$ and $a_i^* = (q_i')^{-1}(([\sum_{k=0}^{p}(-1)^k \mathbf{\Omega}^k]\mathbf{p})_i)$. □

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